

THE BRAUER–MANIN OBSTRUCTION ON KUMMER VARIETIES AND RANKS OF TWISTS OF ABELIAN VARIETIES

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ABSTRACT. Let $r > 0$ be an integer. We present a sufficient condition for an abelian variety A over a number field k to have infinitely many quadratic twists of rank at least r , in terms of the density properties of rational points on the Kummer variety $\text{Km}(A^r)$ of the r -fold product of A with itself.

One consequence of our results is the following. Fix an abelian variety A over k , and suppose that for some $r > 0$ the Brauer–Manin obstruction to weak approximation on the Kummer variety $\text{Km}(A^r)$ is the only one. Then A has a quadratic twist of rank at least r . Hence if the Brauer–Manin obstruction is the only one to weak approximation on all Kummer varieties, then ranks of twists of any positive-dimensional abelian variety are unbounded.

1. INTRODUCTION

1.1. Ranks of twists of abelian varieties. Given an abelian variety A over the number field k , the abelian group $A(k)$ is finitely generated by the Mordell–Weil theorem. Hence there exists an isomorphism

$$A(k) \cong A(k)_{\text{tors}} \oplus \mathbb{Z}^r$$

for some non-negative integer r , which is called the *rank* of $A(k)$, or sometimes simply the rank of A when the ground field k is clear. The rank of an abelian variety is a much-studied invariant, about which many open questions remain.

- (1) For a fixed number field k , and an integer $d > 0$, is the rank of A bounded as A ranges over all d -dimensional abelian varieties over k ?
- (2) For a fixed number field k , and an abelian variety A over k , is the rank of A^c bounded as A^c ranges over all quadratic twists of A ?

The questions above are wide open. For example, it is not known whether there exists an elliptic curve E/\mathbb{Q} of rank at least 29. Honda [6] conjectured that the second question has a positive answer if $\dim(A) = 1$, but this conjecture is not now uniformly believed [10]. We will not answer either of these questions, but we will relate them to another open question concerning the Brauer–Manin obstruction to weak approximation on Kummer varieties.

1.2. Weak approximation. Let X be a smooth, projective, geometrically irreducible variety over a number field k . We denote by $X(\mathbb{A}_k)$ the topological space of adelic points of

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X . By the properness of X , there is a canonical bijection

$$X(\mathbb{A}_k) \xrightarrow{\sim} \prod_{v \in \Omega_k} X(k_v),$$

where Ω_k is the set of places of k . One can ask whether the set $X(k)$ of rational points on X is *dense* in $X(\mathbb{A}_k)$, in other words, whether X satisfies *weak approximation*.

For certain special classes of varieties, the answer to this question is known to be affirmative. For example, this is the case if X is equal to \mathbb{P}_k^n for some integer $n \geq 0$, if X is a quadric hypersurface in \mathbb{P}_k^n , if X is a cubic hypersurface in \mathbb{P}_k^n and $n \geq 16$, if X is the intersection of two quadrics in \mathbb{P}_k^n and $n \geq 8$, or if X is a del Pezzo surface of degree ≥ 5 (see [5] and [7, Theorem 29.4]).

It is certainly not true that weak approximation holds for general (smooth, projective, geometrically integral) varieties X over k . Some counterexamples to weak approximation can be explained by the *Brauer–Manin pairing*. This is a pairing

$$\langle \cdot, \cdot \rangle : X(\mathbb{A}_k) \times \text{Br}(X) \rightarrow \mathbb{Q}/\mathbb{Z},$$

defined using local class field theory (see [15, 5.2]). The pairing $\langle \cdot, \cdot \rangle$ is a homomorphism whenever the first variable is fixed, and a continuous map whenever the second variable is fixed, if \mathbb{Q}/\mathbb{Z} is given the discrete topology. The *Brauer–Manin set* $X(\mathbb{A}_k)^{\text{Br}}$ of X is defined as the left kernel of the Brauer–Manin pairing. It is a closed subset of $X(\mathbb{A}_k)$. It follows from global class field theory that $X(k)$ is contained in $X(\mathbb{A}_k)^{\text{Br}}$, and hence the same is true for its topological closure, denoted $\overline{X(k)}$ (again, see [15, 5.2]).

If $X(\mathbb{A}_k)^{\text{Br}}$ is strictly contained in $X(\mathbb{A}_k)$, then $X(k)$ is not dense in $X(\mathbb{A}_k)$; in this case, we say that there is a *Brauer–Manin obstruction to weak approximation*. If $\overline{X(k)}$ equals $X(\mathbb{A}_k)^{\text{Br}}$, then we say that the Brauer–Manin obstruction to weak approximation on X is *the only one*.

In some cases it is known that the Brauer–Manin obstruction to weak approximation on X is the only one; for example, this is the case if X is a del Pezzo surface of degree 4 such that $X(k) \neq \emptyset$ (see [11]), if $k = \mathbb{Q}$ and X admits a conic bundle structure $X \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ with at least one degenerate fibre and such that all degenerate fibres are defined over \mathbb{Q} (see [2]), or if X is a smooth compactification of a homogeneous space of a connected linear group with connected stabilizers (see [1]). On the other hand, examples of X with $\overline{X(k)} \neq X(\mathbb{A}_k)^{\text{Br}}$ can be found in many places (e.g., see [14]).

Lastly, we mention the following conjecture by Jean-Louis Colliot-Thélène [3]:

Conjecture 1. *Let X be a smooth, projective, geometrically integral variety over a number field k , and assume that $X_{\bar{k}}$ is birationally equivalent with $\mathbb{P}_{\bar{k}}^n$ for some integer n . Then the Brauer–Manin obstruction to weak approximation on X is the only one.*

1.3. Kummer varieties and the Brauer–Manin obstruction. For an abelian variety B over k , we write $\text{Km}(B)$ for its *Kummer variety*, i.e. $\text{Km}(B)$ is the quotient of B by multiplication by -1 ; it is a geometrically integral projective variety. We note that, if $\dim(B) > 1$, then $\text{Km}(B)$ is never smooth. Conjecture 1 suggests that the following question could be a reasonable one.

Question 2. *Let k be a number field and A an abelian variety over k . Let $r > 0$ an integer. Let X be a smooth projective variety birationally equivalent to the Kummer variety $\mathrm{Km}(A^r)$. Is it true that the Brauer–Manin obstruction is the only obstruction to weak approximation on X ?*

In the present paper, we investigate the implications of a positive answer to Question 2 for ranks of twists of abelian varieties over k . In this paper, we will prove:

Theorem 3 (see Corollary 18). *Let k be a number field. Let A be a positive-dimensional abelian variety over k and let $r > 0$ be an integer. Assume that the Brauer–Manin obstruction to weak approximation on X is the only one, where X is some smooth projective model of $\mathrm{Km}(A^r)$. Then A has infinitely many quadratic twists of rank at least r .*

In fact, the conclusion of the theorem holds under the weaker assumption that there exists a density-zero set T of places of k such that $X(k)$ is dense in $\prod_{v \notin T} X(k_v)$ (see Remark 16 and Theorem 17). Hence, if the extension of Conjecture 1 to smooth projective models of Kummer varieties of the form $\mathrm{Km}(A^r)$ were to hold true, then ranks of quadratic twists of any given positive-dimensional abelian variety are unbounded.

2. RATIONAL POINTS ON A KUMMER VARIETY

In this section, we let k be a field of which the characteristic is different from 2. For any abelian variety B over k , we write B_0 for the complement in B of its 2-torsion subscheme. We further denote by $\mathrm{Km}_0(B)$ the quotient of B_0 by its natural μ_2 -action. Since the μ_2 -action on B_0 has no fixed points, it endows B_0 with the structure of μ_2 -torsor over $\mathrm{Km}_0(B)$.

Throughout this section, we fix an abelian variety B . Let \widehat{X} denote $\mathrm{Km}(B)$ and let $q: B \rightarrow \widehat{X}$ denote the quotient map.

2.1. Quadratic twists. We first recall some standard facts about quadratic twists of abelian varieties.

Let L be a field extension of k with separable closure \overline{L} , and let $\Gamma_L = \mathrm{Gal}(\overline{L}/L)$ be the absolute Galois group of L . Let X be a scheme over L . Recall that a scheme X' over L is said to be a *twist* of X if there exists an isomorphism $\phi: X_{\overline{L}} \xrightarrow{\sim} X'_{\overline{L}}$, where the subscripts denote base-change. If X' is a twist of X , and $\phi: X_{\overline{L}} \xrightarrow{\sim} X'_{\overline{L}}$ is an isomorphism, then we may associate with the pair (X', ϕ) the cocycle $c: \Gamma_L \rightarrow \mathrm{Aut}(X_{\overline{L}})$ that maps $\sigma \in \Gamma_L$ to the automorphism $\sigma(\phi^{-1}) \circ \phi$. One says that X' is the twist of X *by the cocycle c* . This defines an injective map from the set of twists of X up to isomorphism to $H^1(L, \mathrm{Aut}(X_{\overline{L}}))$. If X is e.g. a quasi-projective variety, then this map is a bijection. If this is the case, then there is a natural way to associate with a cocycle $c: \Gamma_L \rightarrow \mathrm{Aut}(X_{\overline{L}})$ a pair (X', ϕ) as above (for the last statements, see [12, III.1.3]).

Let c be an element of $H^1(L, \mu_2)$. Since the Γ_L -action on μ_2 is trivial, we have $H^1(L, \mu_2) = \mathrm{Hom}(\Gamma_L, \mu_2) = L^*/L^{*2}$, so we may interpret the elements of $H^1(L, \mu_2)$ as cocycles $\Gamma_L \rightarrow \mu_2$. Then there exists a scheme B^c over L and an isomorphism $\phi: B_{\overline{L}} \xrightarrow{\sim} B^c_{\overline{L}}$ giving rise to the cocycle c . Since $B_{\overline{L}}$ is an abelian variety, we can use ϕ to endow $B^c_{\overline{L}}$ with the structure of an abelian variety over \overline{L} ; it is easy to see that this structure descends to L . Hence B^c is

an abelian variety over L that is the twist of B_L by the cocycle c ; we call B^c the *quadratic twist* of B_L by the cocycle c .

2.2. Quadratic twists and Kummer varieties. Again, we let L be any field extension of k , with separable closure \bar{L} . For all elements c in $H^1(L, \mu_2)$, we consider the pair (B^c, ϕ^c) , where B^c is the quadratic twist of B_L corresponding to c , and $\phi^c: B_{\bar{L}} \xrightarrow{\sim} B_{\bar{L}}^c$ the corresponding isomorphism. We then define a map

$$q^c: B_{\bar{L}}^c \rightarrow \hat{X}_{\bar{L}}$$

by setting $q^c = q \circ (\phi^c)^{-1}$, where we have re-used the letter q to denote the base-change to \bar{L} of the quotient map $q: B \rightarrow \hat{X}$ defined above. It is easily verified that q^c is defined over L ; hence we obtain a morphism

$$q^c: B^c \rightarrow \hat{X}_L.$$

Moreover, since ϕ^c is an isomorphism of abelian varieties, the morphism q^c is again the quotient map for the μ_2 -action on B^c .

Proposition 4. *Let $c \in H^1(k, \mu_2)$ and $d \in H^1(L, \mu_2)$ be such that the restriction of c to $H^1(L, \mu_2)$ equals d . Write B_L^c for the base-change of B^c to L and B_L^d for the twist of B_L by the cocycle d . Then there exists an isomorphism*

$$F_{c,d}: B_L^c \xrightarrow{\sim} B_L^d$$

such that the following diagram is commutative

$$\begin{array}{ccc} B_L^c & \xrightarrow[\cong]{F_{c,d}} & B_L^d \\ & \searrow q^c & \swarrow q^d \\ & \hat{X}_L & \end{array}$$

where \hat{X}_L is the base-change of \hat{X} to L .

Proof. Let $\phi^c: B_{\bar{k}} \xrightarrow{\sim} B_{\bar{k}}^c$ and $\phi^d: B_{\bar{L}} \xrightarrow{\sim} B_{\bar{L}}^d$ be the isomorphisms corresponding to the cocycles c and d , as chosen above. Let $F_{c,d}$ be the \bar{L} -isomorphism

$$\phi^d \circ (\phi^c)^{-1}: B_{\bar{L}}^c \xrightarrow{\sim} B_{\bar{L}}^d.$$

Since $\text{res}_{L/k}(c) = d$, we have that $F_{c,d}$ is defined over L . Finally, we have

$$q_c = q \circ (\phi_c)^{-1} = q \circ \phi_d^{-1} \circ F_{c,d} = q_d \circ F_{c,d},$$

hence the diagram is commutative. \square

2.3. A partition of rational points on a Kummer variety. We write X_0 for $\text{Km}_0(B)$. For each c in $H^1(L, \mu_2)$, we write $q^c: B_0^c \rightarrow X_0$ for the restriction of $q^c: B^c \rightarrow \hat{X}$. As observed above, the μ_2 -action on B_0 endows it with the structure of a μ_2 -torsor over X_0 .

Lemma 5. *Let the notation be as before in this section. The morphisms $q^c: B_0^c \rightarrow X_0$ induce a bijection*

$$(1) \quad \coprod_{c \in H^1(L, \mu_2)} B_0^c(L)/\mu_2(L) \xrightarrow{\sim} X_0(L).$$

Assume that k is a number field and that L is a completion of k . Then $H^1(L, \mu_2)$ is a finite group. If we endow the sets $B_0^c(L)$ and $X_0(L)$ with the topologies coming from the one on L , and the sets $B_0^c(L)/\mu_2(L)$ with the quotient topology, then the bijection (1) is in fact a homeomorphism.

Proof. The first part of the lemma is a consequence of eq. (2.12) of [15] and the discussion leading up to it. We give a sketch of the idea for the sake of completeness. Let $P \in X_0(L)$ be arbitrary. The fibre $q^{-1}(P)$ over P is a μ_2 -torsor over $\text{Spec}(L)$. Such torsors are classified by the Galois cohomology group $H^1(L, \mu_2)$; hence $q^{-1}(P)$ determines an element $c' \in H^1(L, \mu_2)$. For an arbitrary element $c \in H^1(L, \mu_2)$, it is then not hard to see that the fibre $(q^c)^{-1}(P)$ gives a μ_2 -torsor corresponding to the element cc' of $H^1(L, \mu_2)$. Since a torsor over $\text{Spec}(L)$ has an L -point if and only if it corresponds to the identity of $H^1(L, \mu_2)$, we see that $P \in q^c(B_0^c(L))$ if and only if cc' is the identity of $H^1(L, \mu_2)$.

For the second part, we assume that k is a number field and that L is a completion of k . The finiteness of $H^1(L, \mu_2)$ is then a well-known fact. Observe that the maps $B_0^c(L) \rightarrow X_0(L)$ induced by the q^c are continuous, since they are induced by morphisms between projective L -schemes; by the definition of the quotient topology, the maps $B_0^c(L)/\mu_2(L) \rightarrow X_0(L)$ are likewise continuous. This shows that (1) is a continuous bijection. To prove our claim it suffices to show that the map (1) is open; for this, it suffices to show that each of the $B_0^c(L)/\mu_2(L) \rightarrow X_0(L)$ is open. Let $V \subset B_0^c(L)/\mu_2(L)$ be an open subset and let U be its image in $X_0(L)$. The preimage V' of V in $B_0^c(L)$ is open by the continuity of the quotient map $B_0^c(L) \rightarrow B_0^c(L)/\mu_2(L)$, and we have $U = q^c(V')$. Since q^c is an étale morphism, it is open by the implicit function theorem for L . Therefore U is open, which establishes the claim. \square

Remark 6. Assume that $L = \prod_{i=1}^n L_i$ is a finite product of field extensions L_i of k . Then (1) still holds. Indeed, the set $X_0(L)$ is canonically isomorphic to $\prod_{i=1}^n X_0(L_i)$, and the set $B_0^c(L)/\mu_2(L)$ is canonically isomorphic to the product $\prod_{i=1}^n B_0^c(L_i)/\mu_2(L_i)$. If moreover k is a number field and the L_i are completions of k , and we endow the various sets with their natural product topologies, then (1) is again a homeomorphism.

2.4. A useful diagram. We continue the notation from the previous parts of this section. In the diagram of Proposition 7, we will write $H^1(k)$ for $H^1(k, \mu_2)$ and $H^1(L)$ for $H^1(L, \mu_2)$.

Proposition 7. *There exist maps $f_{L/k}$, $\tilde{f}_{L/k}$, and $F_{L/k}$ rendering the following diagram commutative:*

$$\begin{array}{ccc}
X_0(k) & \xrightarrow{\quad} & X_0(L) \\
\uparrow \cong & & \uparrow \cong \\
\coprod_{c \in H^1(k)} B_0^c(k)/\mu_2(k) & \xrightarrow{f_{L/k}} & \coprod_{d \in H^1(L)} B_0^d(L)/\mu_2(L) \\
\downarrow & & \downarrow \\
\coprod_{c \in H^1(k)} B^c(k)/\mu_2(k) & \xrightarrow{\tilde{f}_{L/k}} & \coprod_{d \in H^1(L)} B^d(L)/\mu_2(L) \\
\uparrow & & \uparrow \\
\coprod_{c \in H^1(k)} B^c(k) & \xrightarrow{F_{L/k}} & \coprod_{d \in H^1(L)} B^d(L)
\end{array}$$

Here, the maps labeled \cong are the ones induced by Lemma 5, and the unlabeled maps are the natural inclusion and quotient maps.

Proof. It suffices to define $F_{L/k}$, and check that it induces maps $\tilde{f}_{L/k}$ and $f_{L/k}$ that make the diagram commutative. Let $F_{L/k}$ be the map

$$F_{L/k}: \coprod_{c \in H^1(k, \mu_2)} B^c(k) \rightarrow \coprod_{d \in H^1(L, \mu_2)} B^d(L)$$

that when restricted to $B^c(k)$ equals the map

$$F_{c,d}: B^c(k) \rightarrow B^d(L),$$

where $F_{c,d}$ is as defined in the proof of Proposition 4, and where $d \in H^1(L, \mu_2)$ equals the restriction $\text{res}_{L/k}(c)$ of the cocycle c to L . By passage to the quotient, the map $F_{L/k}$ induces a map $\tilde{f}_{L/k}$ that makes the lower square of the diagram commutative. Likewise, if we let $f_{L/k}$ be the restriction of $\tilde{f}_{L/k}$, it is clear that the middle square is commutative. Finally, it follows from Proposition 4 and the definition of $f_{L/k}$ that the top square is commutative. \square

Remark 8. It follows from Lemma 5 that if $L = \prod_{i=1}^r L_i$ is a finite product of field extensions of k , then Proposition 7 carries over word for word. Now assume additionally that k is a number field, and the L_i are completions of k . Then the maps appearing in the second column of the diagram are continuous, where the sets are endowed with their obvious topologies. By Lemma 5 and Remark 6, the top-right vertical map is a homeomorphism. Furthermore, since the complement of each $B_0^c(L_i)$ in $B^c(L_i)$ is a finite set, it is easy to see that the downward facing map in the second column is the inclusion of an open subset.

3. A CRITERION FOR THE EXISTENCE OF HIGH-RANK TWISTS

We fix a number field k . In this section, we prove a criterion for an abelian variety A to possess a quadratic twist with rank at least r , in terms of density properties of rational points on the Kummer variety of the r -fold product A^r .

Theorem 9. *Let A be an abelian variety over the number field k . Let $r > 0$ be an integer and denote $B = A^r$. Let p be a prime. Let $S \subset H^1(k, \mu_2)$ be a finite subset. Assume L_1, \dots, L_r are pairwise non-isomorphic non-archimedean completions of k , all of good reduction for A , such that*

- (i) *each group $A(L_i)$ has an element P_i of order p ;*
- (ii) *for every quadratic twist A^c of A such that either $c \in S$ or $A^c(k)[p] \neq 0$, we have, for each i , that $A^c(k)$ is contained in $pA^c(L_i)$;*
- (iii) *$\text{Km}_0(B)(k)$ is dense in $\prod_{i=1}^r \text{Km}_0(B)(L_i)$.*

Then there exists $c \in H^1(k, \mu_2)$ such that $c \notin S$ and the rank of A^c is at least r .

The rest of this section is devoted to the proof of Theorem 9. Throughout this section then, let A be an abelian variety over the number field k , let $r > 0$ be an integer, let $B = A^r$ be the r -fold product of A , let p be a prime, and let L_1, \dots, L_r be pairwise non-isomorphic non-archimedean completions of k satisfying conditions (i)–(iii) from Theorem 9. We will write L for the product $\prod_{i=1}^r L_i$.

We need three intermediate lemmas. After Lemma 10, some additional definitions will be made that play a crucial role in the proof.

Lemma 10. *For each i , there exists a surjective group homomorphism $\pi_i: A(L_i) \rightarrow \mathbb{F}_p$.*

Proof. Indeed, by a well-known result about abelian varieties over non-archimedean local fields [8], the group $A(L_i)$ has a finite-index subgroup isomorphic to $W = \mathcal{O}_{L_i}^{\dim(A)}$, where \mathcal{O}_{L_i} denotes the ring of integers of L_i . Since W is torsion-free, the image of P_i in the finite quotient $A(L_i)/W$ must have order p . Clearly then, there exists a surjection $A(L_i)/W \rightarrow \mathbb{F}_p$. This establishes the lemma. \square

Let $V = \mathbb{F}_p^r$. Let the group $\mu_2(L) = \prod_{i=1}^r \mu_2(L_i)$ act on V in such a way that the i -th basis element $m_i = (1, \dots, -1, \dots, 1)$ of $\mu_2(L)$, with -1 at the i -th coordinate, multiplies the i -th component of every vector in V by -1 , while leaving the other components unchanged. We denote by $\pi' = (\pi_1, \dots, \pi_r)$ the combined map from $A(L) = \prod_{i=1}^r A(L_i)$ to V . Note that the map π' is $\mu_2(L)$ -equivariant for the natural $\mu_2(L)$ -action on $A(L)$. Taking into account that $B(L) = A(L)^r$, the map π' induces a map

$$\pi: B(L) \rightarrow V^r.$$

We endow V^r with the “diagonal” $\mu_2(L)$ -action: regarding the elements of V as column vectors and the elements of V^r as r -by- r -matrices, the basis vector $e_i \in \mu_2(L)$ acts on an element of V^r by multiplying its i -th row by -1 . The map π is then $\mu_2(L)$ -equivariant. Finally, define $\Delta \subset V^r$ as the subset containing the $v \in V^r$ having determinant zero.

Lemma 11. *The image of Δ under the quotient map $V^r \rightarrow V^r/\mu_2(L)$ is a proper subset of $V^r/\mu_2(L)$.*

Proof. The $\mu_2(L)$ -action on V^r sends determinant-zero matrices to determinant-zero matrices. Hence the image of Δ in $V^r/\mu_2(L)$ consists precisely of the orbits of determinant-zero matrices, which of course form a proper subset of $V^r/\mu_2(L)$. \square

Lemma 12. *Let c be an element of $H^1(k, \mu_2)$ such that for all $i \in \{1, 2, \dots, r\}$ we have that $\text{res}_{L_i/k}(c)$ is the identity of $H^1(L_i, \mu_2)$. Assume that either $c \in S$ or the rank of $A^c(k)$ is less than r . We claim: the image of the composition of group homomorphisms*

$$B^c(k) \rightarrow B^c(L) \rightarrow B(L) \xrightarrow{\pi} V^r,$$

is contained in Δ . Here, the first arrow is the natural inclusion, and the second arrow is induced by the isomorphisms $F_{c,1}^{(i)}: B_{L_i}^c \xrightarrow{\sim} B_{L_i}$ provided by Proposition 4.

Proof. First, suppose that c is such that $c \in S$ or $A^c(k)[p] \neq 0$. Then by assumption (ii) of Theorem 9, the image of $B^c(k)$ is contained in $pB^c(L)$, and therefore its image in V^r is identically 0. We may therefore suppose that c is not contained in S , so that the rank of A^c is less than r , and that $A^c(k)[p] = 0$. Let $Q = (Q_1, \dots, Q_r) \in A^c(k)^r$, and let $\overline{Q} = (\overline{Q}_1, \dots, \overline{Q}_r)$ denote its image in V^r , so that the \overline{Q}_i are elements of V such that $\overline{Q}_i = \pi_i(Q_i)$ for each i . Since the rank of $A^c(k)$ is at most $r - 1$, we have $a_1 Q_1 + a_2 Q_2 + \dots + a_r Q_r = 0$ for some integers a_1, a_2, \dots, a_r . Since $A^c(k)[p] = 0$, we may assume that the a_i are not all divisible by p . Then we must have a non-trivial relation $a_1 \overline{Q}_1 + a_2 \overline{Q}_2 + \dots + a_r \overline{Q}_r = 0$, in V , showing that $\overline{Q} \in \Delta$. \square

Proof of Theorem 9. We assume that, for all $c \notin S$, the rank of A^c is less than r . Let $H^1(k)_1 \subset H^1(k, \mu_2)$ be the subset consisting of those $c \in H^1(k, \mu_2)$ whose restriction to each of the L_i is the identity. We then consider the following diagram.

$$\begin{array}{ccccc} \coprod_{c \in H^1(k)_1} B^c(k)/\mu_2(k) & \xrightarrow{\tilde{f}} & B(L)/\mu_2(L) & \xrightarrow{\overline{\pi}} & V^r/\mu_2(L) \\ \uparrow & & \uparrow & & \uparrow \\ \coprod_{c \in H^1(k)_1} B^c(k) & \longrightarrow & B(L) & \xrightarrow{\pi} & V^r \end{array}$$

The maps forming the left-hand square are induced by those from the diagram of Proposition 7. It was shown there that the left-hand square is commutative. The map $\pi: B(L) \rightarrow V^r$ was defined earlier in the proof, and it was observed to be $\mu_2(L)$ -equivariant. We define $\overline{\pi}: B(L)/\mu_2(L) \rightarrow V^r/\mu_2(L)$ as the map induced by π , while we let the right-hand vertical map be the quotient map; with these choices, it is clear that the entire diagram is commutative. We consider V^r and $V^r/\mu_2(L)$ as topological spaces endowed with the discrete topology. Then π is continuous by the fact that the topology on $B(L)$ is the profinite one, while $\overline{\pi}$ is continuous by the quotient property of $B(L)/\mu_2(L)$.

By Lemma 12, the composite map

$$(2) \quad \coprod_{c \in H^1(k)_1} B^c(k)/\mu_2(k) \rightarrow V^r/\mu_2(L)$$

factors via $\Delta/\mu_2(L)$. By Lemma 11, the image of the map (2) is a proper subset of $V^r/\mu_2(L)$. Since $\bar{\pi}$ is a continuous surjection and $V^r/\mu_2(L)$ is discrete, this means that the image of \tilde{f} does not lie dense. But then the image of the map $f_{L/k}$ (as in the diagram from Proposition 7) does not lie dense. Again by the diagram of Proposition 7, this shows that $X_0(k)$ cannot be dense in $X_0(L)$, which is a contradiction. \square

4. BRAUER GROUPS OF KUMMER VARIETIES OVER NUMBER FIELDS

In this section, we let k be a number field. Let B be an abelian variety over k . Let $\hat{X} = \text{Km}(B)$ be its Kummer variety and let X be any smooth projective model of \hat{X} . We denote by $\text{Br}(X)$ the Brauer group of X , and by $\text{Br}_0(X)$ the image in $\text{Br}(X)$ of $\text{Br}(k)$.

Proposition 13. *The quotient $\text{Br}(X)/\text{Br}_0(X)$ is finite.*

Proof. Let $\text{Br}_1(X) = \ker(\text{Br}(X) \rightarrow \text{Br}(X_{\bar{k}}))$ denote the subgroup of algebraic elements in $\text{Br}(X)$. (As usual, we write $X_{\bar{k}}$ for the base-change of X to \bar{k} .) It suffices to prove that the quotients $\text{Br}_1(X)/\text{Br}_0(X)$ and $\text{Br}(X)/\text{Br}_1(X)$ are finite.

By the Hochschild–Serre spectral sequence, the quotient $\text{Br}_1(X)/\text{Br}_0(X)$ is isomorphic to the Galois cohomology group $H^1(k, \text{Pic}(X_{\bar{k}}))$. Since the topological fundamental group $\pi_1(X(\mathbb{C}))$ vanishes [17], the variety X has trivial Albanese variety, so that $H^1(k, \text{Pic}(X_{\bar{k}})) = H^1(k, \text{NS}(X_{\bar{k}}))$. Again since $\pi_1(X(\mathbb{C}))$ vanishes, the Néron–Severi group of $X_{\bar{k}}$ is torsion-free, which implies that $\text{NS}(X_{\bar{k}})$ is a finitely generated torsion-free abelian group. This implies that $H^1(k, \text{NS}(X_{\bar{k}}))$, and therefore $H^1(k, \text{Pic}(X_{\bar{k}}))$, are both finite.

Write $\Gamma = \text{Gal}(\bar{k}/k)$. The group $\text{Br}(X)/\text{Br}_1(X)$ embeds into $\text{Br}(X_{\bar{k}})^\Gamma$. Let res be the restriction map from $\text{Br}(X_{\bar{k}})^\Gamma$ to $\text{Br}(B_{\bar{k}})^\Gamma$, where $B_{\bar{k}}$ is the base-change of B to \bar{k} . The group $\text{Br}(B_{\bar{k}})^\Gamma$ is finite by [16, Theorem 1.1(i)]; hence the map res has finite image. The elements of the kernel of res have order at most 2 by a standard restriction-corestriction argument. The group $\text{Br}(X_{\bar{k}})$ is isomorphic to the direct sum of $(\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$ and a finite abelian group [4, eq. (7)], where b_2 is the second Betti number of $X_{\bar{k}}$ and ρ is the Picard number of $X_{\bar{k}}$; therefore $\text{Br}(X_{\bar{k}})$ has only finitely many elements of order at most 2. It follows that the kernel of res is finite. Since its image was already shown to be finite, we must have that $\text{Br}(X_{\bar{k}})^\Gamma$ is finite. Since $\text{Br}(X)/\text{Br}_1(X)$ injects into $\text{Br}(X_{\bar{k}})^\Gamma$, it too is finite, and so we are done. \square

Proposition 14. *If X is a smooth projective model of the Kummer variety of an abelian variety B over k , then $X(\mathbb{A}_k)^{\text{Br}}$ is a non-empty open and closed subset of $X(\mathbb{A}_k)$.*

Proof. Observe that $B(k)$ is non-empty since it contains 0. The Lang–Nishimura Lemma applied to the rational map $B \dashrightarrow X$ between smooth and proper varieties then implies that $X(k)$ is non-empty. The subset $X(\mathbb{A}_k)^{\text{Br}}$ is therefore also non-empty, for it contains $X(k)$. It is open and closed, since it is the intersection of the open and closed subsets

$$X(\mathbb{A}_k)^{\mathcal{A}} = \{(x_v)_v \in X(\mathbb{A}_k) : \langle (x_v)_v, \mathcal{A} \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ is the Brauer–Manin pairing, and where \mathcal{A} runs through a set of coset representatives of the group quotient $\text{Br}(X)/\text{Br}_0(X)$, which is finite by Proposition 13. \square

5. THE BRAUER–MANIN OBSTRUCTION AND RANKS OF TWISTS

We fix again a number field k .

Lemma 15. *Let A be an abelian variety over k , and let T be any finite set of places of the number field k . Let $S' \subset H^1(k, \mu_2)$ be a finite subset. For every prime p and every integer $r > 0$, there exist pairwise non-isomorphic non-archimedean completions L_1, \dots, L_r of k , none of which arise from any place in T , such that for each $i \in \{1, 2, \dots, r\}$ we have:*

- (i) *the group $A(L_i)$ contains a point of order p ;*
- (ii) *for all $c \in S'$ we have $A^c(k) \subset pA^c(L_i)$.*

Proof. It follows from the finiteness of S' and from the Mordell–Weil theorem that there exists a number field $\ell \supset k$ such that $A(\ell)[p] \neq 0$ and such that for all $c \in S'$ we have $A^c(k) \subset pA^c(\ell)$. The Čebotarev density theorem implies that the set of places of k splitting completely in ℓ has positive density, and the lemma follows. \square

Remark 16. The proof of Lemma 15 shows that we could have relaxed the assumption that T is finite to T being merely of density zero. We will not make use of this fact, however.

Theorem 17. *Let A be an abelian variety over the number field k , and let $r > 0$ be an integer. Write $B = A^r$. Let X be a smooth projective variety birationally equivalent to $\text{Km}(B)$. Suppose there exists a finite set $T \subset \Omega_k$ such that, for all r -element subsets T' of $\Omega_k \setminus T$, we have that $X(k)$ is dense in $\prod_{v \in T'} X(k_v)$, when the latter set is equipped with the product topology. Then A has infinitely many twists of rank at least r .*

Proof. Let $S \subset H^1(k, \mu_2)$ be any finite subset. It suffices to show that there exists $c \notin S$ such that the rank of A^c is at least r .

By the main result of [13], the number of $c \in H^1(k, \mu_2)$ with $A^c(k)_{\text{tors}} \neq A^c(k)[2]$ is finite. Fix a prime $p > 2$. Then Lemma 15 yields the existence of pairwise non-isomorphic non-archimedean completions L_1, \dots, L_r of k , none of which arise from T and all of which are of good reduction for A , such that for $i \in \{1, 2, \dots, r\}$ we have:

- (i) the group $A(L_i)$ contains a point P_i of order p ;
- (ii) for all c such that either $c \in S$ or $A^c(k)[p] \neq 0$, we have $A^c(k) \subset pA^c(L_i)$.

We write $L = \prod_{i=1}^r L_i$. By the assumptions of the theorem, the set $X(k)$ is dense in the topological space $X(L)$. By [9, Proposition 3.7], the density of $X(k)$ in $X(L)$ implies the density of $\text{Km}_0(B)(k)$ in $\text{Km}_0(B)(L)$. Therefore the conditions of Theorem 9 are satisfied. By Theorem 9, there exists $c \notin S$ such that the rank of A^c is at least r , which proves the theorem. \square

Corollary 18. *Let k be a number field. Let A be a positive-dimensional abelian variety over k and let $r > 0$ be an integer. Assume that the Brauer–Manin obstruction to weak approximation on X is the only one, where X is some smooth projective model of $\text{Km}(A^r)$. Then A has infinitely many quadratic twists of rank at least r .*

Proof. Indeed, suppose that the hypothesis of the corollary is satisfied. Then by Proposition 14 we have that for all abelian varieties A over k , if X is a smooth projective variety birationally equivalent to $\text{Km}(A^r)$ for some integer $r > 0$, then $X(k)$ is dense in $\prod_{v \notin T} X(k_v)$ for some finite subset T of Ω_k . The result then follows from Theorem 17. \square

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